Social Positions Analysis

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• Chapter 12-15 of “Introduction to social network methods”
Social Positions

• **Social Positions/Roles**
  – Actors who are connected in *the same* way to the rest of the network are said to be *equivalent* and to occupy the *same position*

• **Objective**
  – To partition actors into *mutually exclusive* classes of *equivalent* actors who have similar relational patterns
Social Positions

• Three classes (levels) to find equivalent positions/roles
  – Structural equivalence (SE)
  – Automorphic equivalence (AE)
  – Regular equivalence (RE)
Structural Equivalence (1/3)

• **Structurally equivalent (S.E.)**
  – Two nodes have identical neighbors

[Def] Nodes $a$ and $b$ are **structurally equivalent** if, whenever $\{a,x\}$ is an edge of $G$, then so is $\{b,x\}$, and conversely ($x \neq a, b$)
Bill and Joe are structurally equivalent actors

(a) Simple network

(b) Bill's ego-network

(c) Joe's ego network

FIGURE 1.
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More Example

A B C D E F G H

| A | 0 1 0 0 1 0 0 1 |
| B | 1 0 1 0 0 1 1 0 |
| C | 0 1 0 1 1 0 0 1 |
| D | 0 0 1 0 0 0 1 0 |
| E | 1 0 1 0 0 1 1 0 |
| F | 0 1 0 0 1 0 0 1 |
| G | 0 1 0 1 1 0 0 1 |
| H | 1 0 1 0 0 1 1 0 |

Rearrange by rows

A B C D E F G H

| A | 0 1 0 0 1 0 0 1 |
| F | 0 1 0 0 1 0 0 1 |
| C | 0 1 0 1 1 0 0 1 |
| G | 0 1 0 1 1 0 0 1 |
| B | 1 0 1 0 0 1 1 0 |
| E | 1 0 1 0 0 1 1 0 |
| H | 1 0 1 0 0 1 1 0 |
| D | 0 0 1 0 0 0 1 0 |
Structural Equivalence in Directed Graph

- Node $a$ and $b$ are structurally equivalent if,
  
  1. Whenever $(a,x)$ is an arc in $G$ so is $(b,x)$ and conversely,
     
     and
  
  2. Whenever $(x,a)$ is an arc in $G$ so is $(x,b)$ and conversely
Structural Equivalence in multi-relational graph

• Nodes are structurally equivalent if they are structurally equivalent in each of the constituent graphs
SE vs. Graph-theoretic Measures

- Nodes that are structurally equivalent
- Nodes have **identical graph-theoretic measures** such as centrality, degree, clustering coefficient, etc.

E.g. a, h are identical (graph-theoretic measure) but not structurally equivalent (different neighborhoods)
Limitations of Structural Equivalence

• Need for identical ties
  – i.e., for two managers to be structurally equivalent, they would need to oversee the same workers

• It is too strict a measure to identify even slightly different actors
  – i.e. managers in two companies

• We need more generalized measures of equivalence
Graph Isomorphism

• Two graphs G and H are isomorphic iff:

\[ f : V(G) \rightarrow V(H) \]

any two adjacent nodes u and v from G can be mapped to two adjacent nodes f(u) and f(v) in H

\[ f(a) = 1 \]
\[ f(b) = 6 \]
\[ f(c) = 8 \]
\[ f(d) = 3 \]
\[ f(g) = 5 \]
\[ f(h) = 2 \]
\[ f(i) = 4 \]
\[ f(j) = 7 \]
Graph Isomorphism: example

• A **one-to-one mapping** of one set of objects to another such that the relationships among the objects are also preserved

• The only possible differences between isomorphic graphs are the labels of the nodes (if any)

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**FIGURE 5.** Graphs $G$ and $H$ are isomorphic.

**TABLE 1**

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**FIGURE 6.** Graph $K$ is not isomorphic with any graph in Figure 5.
Automorphic Equivalence (AE)

[Def] Nodes $a$ and $b$ in a graph $G$ are automorphically equivalent if, after exchanging the label of $a$ and $b$, it is possible to re-label some of the rest of the vertices to form an isomorphic graph to $G$.

\[
\begin{align*}
\alpha(1) &= 2 \\
\alpha(2) &= 1 \\
\alpha(3) &= 4 \\
\alpha(4) &= 3 \\
\alpha(5) &= 6 \\
\alpha(6) &= 5 \\
\alpha(7) &= 7
\end{align*}
\]
Dual Definition of AE

• Actors are automorphically equivalent if we can permute the graph in such a way that exchanging the two actors has no effect on the distances among all actors in the graph.
An Intuitive Story

• Suppose there are 10 workers in the 7/11 restaurant, who report to one manager.
  – The manager reports to a franchise owner.
  – The franchise owner also controls the FAMILY store. It too has a manager and 10 workers.

• Now, if the owner decides to transfer the manager from 7/11 to FAMILY (and vice versa), the network would have been disrupted.

• But if the owner transfers both the managers and the workers to the other restaurant, all of the network relations remain intact.

• Transferring both the workers and the managers is a permutation of the graph that leaves all of the distances among the pairs of actors exactly as it was before the transfer.
An Illustration

Automorphic Equivalent Sets:
{a, c, h, j}
{b, d, g, i}
{e, f}

Structural Equivalent Sets:
{a, c}
{b, d}
{g, i}
{h, j}
Find AE Set: Using Neighborhood Degree Distribution (NDD)

1. For each node, finding its NDD.
2. Copy the graph into two.
3. Randomly pairing two unpaired nodes \((u,v)\) from each group with identical “neighborhood degree distribution” (NDD)
4. If the any of the neighborhood nodes of \((u,v)\) are not paired
   - finding one node from each neighborhood \((u', v')\) that has identical NDD to pair (cannot use nodes that are already paired).
     • If fail to find a pair \(\rightarrow\) go back to 3
5. If all nodes are paired \(\rightarrow\) congratulations you find an AE set
6. Let \((u,v)=(u',v')\) and goto 4
Find Automorphically Equivalent Set

• The profile similarity method

1. Calculate **all-pair geodesic distance** matrix $M$, in which $m_{ij} = dist_{ij}$

2. Sort each row vector by ascending order

3. Perform **clustering by computing the pair-wise distance** for vectors
An Example

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all-pair geodesic distance matrix

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0 1 1 1 2 2 2 2 3 3
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Ascending ordered vectors

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Clustering by pair-wise distance

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AE vs. SE

• More general than Structural Equivalence
  – Structurally Equivalent actors are always Automorphically Equivalent, however the opposite is not true

• Automorphic equivalence is a global view to the whole network while local view for structural equivalence

• A.E. identifies the way in which nodes are connected to others while S.E. identifies who an actor is connected to

• Nodes with the same automorphic equivalence may be adjacent, distant, or completely unreachable
Potential Problem for AE

• Sometimes the quantity doesn’t matter that much
  – E.g., a parent with 2 children does not play the same role as one with 3 children
Regular Equivalence

• Def: Two actors are equivalent if they have the same relations with equivalent others

• E.g. X calls American airlines and talk to clerk about booking flight, while Y do the same with their clerk
  – X can Y are equivalent because the clerks are in the equivalent class (while these clerks are equivalent because X and Y are of equivalent class)
Intuitive Story of RE

• Consider two men
  – Each has children
    • Though they have different numbers of children, and, obviously have different children
  – Each has a wife
    • Though again, usually different persons fill this role with respect to each man
  – Each wife, in turn also has children and a husband
    • I.e., they have ties with one or more members of each of those sets
  – Each child has ties to one or more members of the set of “husbands” and “wives”
Regular Equivalence (1/3)

[Def] Two nodes $a$ and $b$ are regular equivalent if, whenever $(a,x)$ is an edge, then there is some node $y$ that is regularly equivalent to $x$ for which $(b,y)$ is an edge.
Regular Equivalence (2/3)

- Regularly equivalent nodes are not necessarily connected to the same third parties, but they are connected to equivalent third parties (though not necessarily in the same quantity)
**Maximal Regular Equivalence**

- A graph may have multiple colorations that are regular, especially undirected graphs.
- We usually consider **maximal** regular equivalence.
Equivalence Review (1/3)

• **Structural equivalence**
  
  – Two actors have the same type of ties to **the same** people
Equivalence Review (2/3)

- Automorphic equivalence
  - Nodes are equivalent with respect to all graph theoretic properties
Equivalence Review (3/3)

- Regular equivalence
  - Perhaps multiple R.E. partitions in a network, we tend to find the **maximal** regular equivalence position, the one with the **fewest positions**

![Diagram of management hierarchy with top boss, manager of 3 different stores, and workers.]

Top boss

Manager of 3 diff. stores

Workers